Quantum Optics
Lecture 8

Topics: – cavity QED basics & spont. emission enhancement
- QND measurements & dispersive regime of cQED.
- cavity input & output relations (theory)
- measurement backaction.

DISPERSIVE REGIME OF CAVITY QED :

(Scully 19.3)

$$\hat{H} = \frac{\hbar\omega_{12}}{2}\hat{\sigma}_z + \hbar\omega a^+a + \hbar g(\hat{\sigma}^+ a + \hat{\sigma}\hat{a}^+)$$

The Hamiltonian expressed in the $|1\rangle\otimes|n+1\rangle$; $|2\rangle\otimes|n\rangle$ basis:

$$\hat{H} = \begin{pmatrix} -\hbar\omega_{12}/2 + n\hbar\omega & \hbar g\cdot\sqrt{n+1} \\ \hbar g\cdot\sqrt{n+1} & +\frac{\hbar\omega_{12}}{2}+n\hbar\omega \end{pmatrix}$$

Note that off diagonal terms are equal. $\langle 1,n+1|\,\sigma a^+|2,n\rangle = \sqrt{n+1}$
$\langle 2,n|\,\sigma^+ a\,|1,n+1\rangle = \sqrt{n+1}$

Matrix can be diagonalized: Eigenstates:

$$\boxed{|+\rangle_n = \underbrace{\frac{2g\sqrt{n+1}}{(\Omega_n-\Delta)+4g^2(n+1)}}_{=\cos\theta_n}|2,n\rangle - \underbrace{\frac{(\Omega_n-\Delta)}{(\Omega-\Delta)^2+4g^2(n+1)}}_{=\sin\theta_n}|1,n+1\rangle}$$

$$\boxed{\Omega_n = \sqrt{\Delta^2+4g^2(n+1)}}$$ (ie $\Omega_{n=0} = \sqrt{4g^2} = 2g$

$$|-\rangle_n = \sin\theta_n\,|2,n\rangle + \cos\theta_n\,|1,n+1\rangle$$

→ resonant case $\omega_{12}=\omega$
for $\Delta = 0$
this is entangled state
$|\pm,0\rangle = (|1_A,0_P\rangle \pm |2_{Atom},0_{Ph}\rangle)/\sqrt{2}$
VACUUM RABI SPLITTING.

The Energies are given by: (Δ: detuning)

$$E_+ = \hbar\omega\cdot n + \hbar\omega_{12}/2 - \frac{\hbar}{2}(\Omega_n-\Delta)$$
$$E_- = \hbar\omega(n+1) - \hbar\omega_{12}/2 + \underbrace{\frac{\hbar}{2}(\Omega_n-\Delta)}_{\text{Plotted lightshift}}$$

The degeneracy of the two manifolds is lifted.

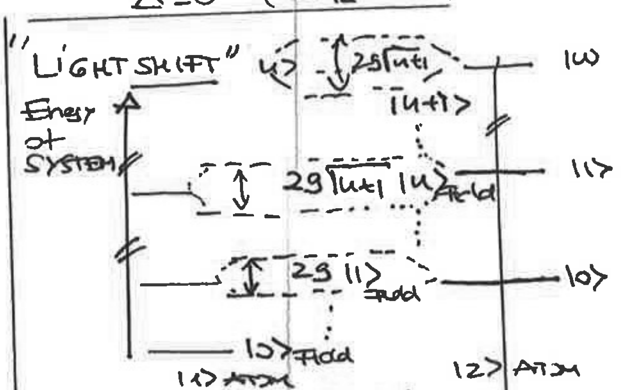
ie $\det(\hat{H} - E^{\pm}\mathbb{1}) \overset{!}{=} 0$ ie eigenvalues.

$\Delta = 0$ ($\omega_{12} = \omega$)

Thus

initially degenerate levels

E^+_{n} —— $+\hbar/2(\Omega_n-\Delta)$
$[\hbar\omega n] + \hbar\omega_{12}/2$
E^-_{n} —— $-\hbar/2(\Omega_n-\Delta)$
$[\hbar\omega(n+1) - \hbar\omega_{12}/2]$



cavity modes coupled to TLS (atom, qubit..)

very important regime: $\Delta \gg 2g\sqrt{n} = \Omega_n$

Thus $|+\rangle \simeq |1,n+1\rangle$
$|-\rangle \simeq |2,n\rangle$

Graphical presentation of the resonant case of cQED.

Dispersive regime of CQED: consider

$$\boxed{\begin{aligned} E_{+n} &= \hbar\omega(n-1) + \omega_{12}\hbar/2 - \cancel{\hbar/2\ldots} - \hbar/2(\Omega_n - \Delta) \\ E_{-n} &= \hbar\omega(n) - \omega_{12}\hbar/2 + \hbar/2(\Omega_n - \Delta) \end{aligned}}$$ EXACT diagonalization

yield in the $\Delta \gg g^2$ regime

$$\boxed{\begin{aligned} E_{+n} &= \hbar\omega n + \tfrac{1}{2}\omega_{12}\hbar - \frac{\hbar g^2(n+1)}{\Delta} \\ E_{-n} &= \hbar\omega(n+1) - \tfrac{1}{2}\hbar\omega_{12} + \hbar\frac{g^2(n+1)}{\Delta} \end{aligned}}$$ Total Energy of System in diagonalized state

Since: $\Omega_n \equiv \sqrt{4g^2(n+1)+\Delta^2} = \Delta\sqrt{\frac{4g^2(n+1)}{\Delta^2}+1}$ ie $\frac{4g^2(n+1)}{\Delta^2} \ll 1$

$$\frac{\hbar}{2}(\Omega_n - \Delta) \simeq \frac{\hbar}{2}\left[\Delta\left(\sqrt{1+\frac{4g^2(n+1)}{\Delta^2}}\right) - \Delta\right] = \frac{\hbar}{2}\,\Delta\frac{4g^2(n+1)}{\Delta^2\,2} = \boxed{\hbar\frac{g^2(n+1)}{\Delta}}$$ correction to energy

& $\boxed{|+\rangle \simeq |2,n\rangle \quad |-\rangle \cong |1,n+1\rangle}$

Thus: the shift is $\boxed{\hbar\delta_n = \frac{\hbar g^2(n+1)}{\Delta}}$ Photon number dependent

Note that $E_{1,0} = -\frac{\hbar}{2}\Delta$

$\boxed{\text{The shift per atom is } \left(\frac{\hbar g^2}{\Delta}\right)(n+1/2) \text{ or cavity changed by } (\hbar g^2/\Delta)}$

KEY POINT: the shift is proportional to photon number (n):

$$\boxed{\hat{H}_{int} = \frac{\hbar g^2}{\Delta}(\hat{a}^+\hat{a})\,\hat{\sigma}_z}$$

$\hat\sigma_z = \hat\sigma_+\hat\sigma_-$

(homework) note that $\hat{H}$ obtained by unitary Transformation: $U \equiv e^{g/\Delta(a\sigma^+ - a^+\sigma^-)}$

$UHU^+ \underset{\Delta\gg g}{\simeq} (\hbar\omega_r + \frac{g^2}{\Delta}\sigma_z)a^+a + \frac{\hbar}{2}(\Omega + \frac{g^2}{\Delta})\sigma_z$

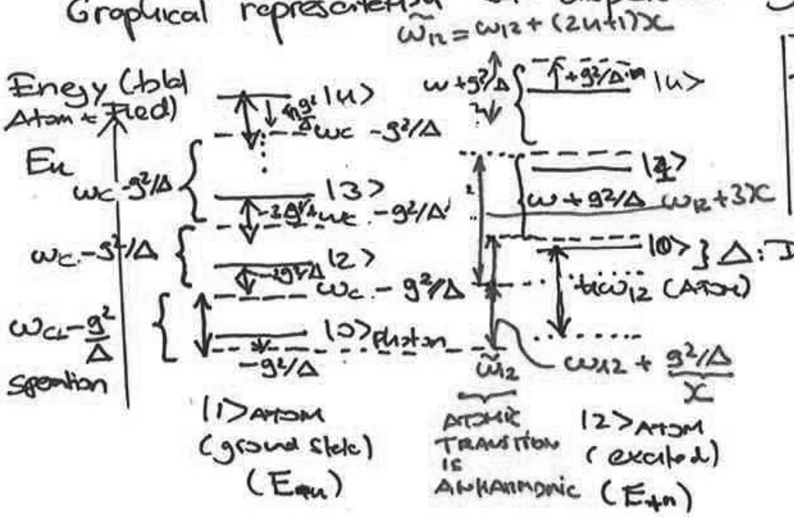
Thus the frequency of qubit is pulled!

This Hamiltonian has thus special properties. Consider the two level system Energy $\hat{H}_0|1\rangle = -\hbar\omega_{12}/2$ $\hat{H}_0|2\rangle = +\hbar\omega_{12}/2$ (bare freq).

$$\frac{E_2 - E_1}{\hbar} \triangleq \omega_{12} = \hbar\omega_{12}\left(1 + n\frac{g^2}{\Delta}\hat{a}^+\hat{a}\right)$$

Graphical representation of dispersive regime. $x \equiv g^2/\Delta$

$\tilde{\omega}_{12} = \omega_{12} + (2n+1)x$



light/photon thus the levels are equally spaced by $\omega_{12} \mp g^2/\Delta$ but depend on excited or ground state!

ie $\omega - \omega_{12}$ arising from $|E_{+n} - E_{-n}|$

$E_{+n} = \hbar\omega_{12} + \hbar\omega$

$E_{-n} =$

$$\boxed{E_{\pm n} = (n+1)\underbrace{\hbar\omega}_{\text{Light}} \pm \frac{\hbar}{2}\sqrt{4g^2(n+1)+\Delta^2}}$$

Exact expression

ie we have a nonlinear pulling of the ~~atomic~~ photon energy (ie frequency) that is NONLINEAR at the SINGLE PHOTON LADDER. or:

re-express states: $|+\rangle \simeq |2,n\rangle$ and $|-\rangle = |1,n+1\rangle$ in basis: (original one ATOM ⊗ FIELD)

$\boxed{|+,0\rangle \simeq |2,1\rangle - g/\Delta\cdot|2,0\rangle}$ $\boxed{|-,0\rangle = +\frac{g}{\Delta}|n_A,1\rangle + |2_A,0\rangle}$

admixing of state

Quantum Optics — Lecture 8 — DISPERSIVE REGIME

The dispersive regime thus causes a light shift that is state dependent ($|1\rangle$ or $|2\rangle$ in ATOMIC BASIS) → readout of qubits in a QND manner (more later)

Hence the ATOMIC TRANSITION and PHOTON Transition frequencies are AC STARK / LAMB shifted.

Photon's: energy shift $E^{\text{shift}}_{\pm,n} = \underbrace{\hbar\omega n}_{\text{ie Bosonic, equally spaced.}} \pm g^2/\Delta n$ (ie Qubit / ATOM dependent resonance shift)

Atom frequency shift (ie Transition frequency) $E_{TS} = \pm\frac{\hbar\omega_{12}}{2} + \underbrace{\frac{g^2}{\Delta} n}_{\text{Photon number dependent transition}}$

The behavior can be reproduced by performing a transformation of the Hamiltonian (unitary)

$$\hat{U} = \exp\left(\frac{g}{\Delta}\hat{a}\hat{\sigma}^+ - \hat{a}^+\hat{\sigma}^-\right)$$

and

$$\boxed{U H U^+ \underset{\text{2nd order}}{\approx} \underbrace{\hbar\omega \hat{a}^+\hat{a}}_{\text{photon}} + \hbar\omega\frac{g^2}{\Delta}\hat{\sigma}_z + \underbrace{\frac{\hbar}{2}\omega_{12}\hat{\sigma}_z}_{\text{ATOM}} + \underbrace{\frac{g^2}{\Delta}\hat{\sigma}_z}_{\text{shift in level (static)}}}$$

regrouping

$$\hat{H} = \hbar\omega a^+a\left(1 + \frac{g^2}{\Delta}\sigma_z\right) + \left(\frac{\hbar\omega_{12}}{2} + \frac{g^2}{\Delta}\right)\hat{\sigma}_z$$

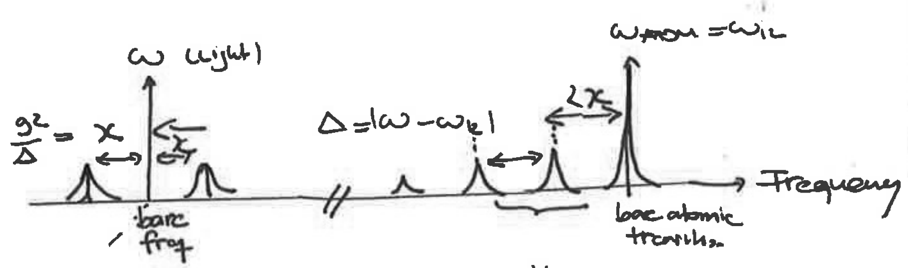
$\left(\pm\frac{g^2}{\Delta}\right)$ shift — AC Stark shift of photon (ie cavity)

or:

$$\hat{H} = \hbar\omega a^+a + \hat{\sigma}_z\left(\frac{\hbar\omega_{12}}{2} + \frac{g^2}{\Delta} + \underbrace{\frac{g^2}{\Delta}a^+a\hbar\omega}_{\text{shift of the ATOM depending on } n}\right)$$

$\frac{n g^2}{\Delta}$ shift

Frequency Landscape:



shift is $2\chi \cdot n$ for n-photons

Ref. · Haroche book / lectures
· Scully Quantum Optics.

Quantum Optics
Lecture

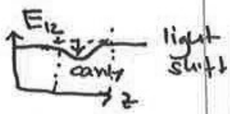
Dispersive shift for QND photon measurements

Send Atom through cavity in superposition state (use ATOM's to probe LIGHT)

→ Ramsey interferometer "Method of separated fields"

$|\psi\rangle_R = \frac{1}{\sqrt{2}}(|e\rangle \pm |g\rangle)$ used in any atomic clock

meterstates: Projection to $|1\rangle\langle 1|$
or $|2\rangle\langle 2|$
ie excited or ground state.



Note $\sim\left(n + \frac{g^2}{\Delta}\right)$ is atomic phase shift of atom
→

Likewise the field shifts by $\left(\frac{g^2}{\Delta}\right)$ (single photon Kerr shift)

ie the light shift per photon ie energy conservation!

ie cav frequency becomes $\omega_{car} = \omega - g^2/\Delta$ ie frequency is pulled!
shift is state dependent $|e\rangle, |g\rangle$
→ +/- sign change

→ Kerr shift due to ATOM.

NOTE: QUANTUM GATE ie phase shift of optical field is state dependent.

consider evolution in cavity for time T_{car}.

evolution:

(1) Ramsey zone $\left| \begin{array}{l} C_{g,n} = i\Omega_R/2 \cdot C_{e,n} \\ C_{e,n} = i\Omega_R/2 \cdot C_{g,n} \end{array} \right|$ → $\begin{array}{l} C_{g,n} = \cos(\frac{\Omega_R t}{2}) \\ C_{e,n} = \sin(\frac{\Omega_R t}{2})\, i \end{array}$

(2) when passing through cavity $\begin{array}{l} C_{g,n} = \cos(\frac{\Omega_R t}{2}) \times \underbrace{\left[\exp(i \frac{g^2 n}{\Delta} \cdot \tau)\right]}_{\text{Phase shift}} \\ C_{e,n} = i \sin(\Omega_R t/2) \end{array}$

carry

(3) second interferometer. $\boxed{\begin{array}{l} C_{g,n} = \cos^2(\Omega_R t/2) \exp(\frac{i g^2 n}{\Delta}\tau) - \sin^2(\Omega_R T/2) \\ C_{e,n} = \frac{i}{2} \sin(\Omega_R t) \left[\exp(i g^2 \frac{n}{\Delta} \cdot \tau) + 1\right] \end{array}}$

Therefore for $\Omega_R t = \pi/2$ one obtains

$$P_g = |c_{g,n}|^2 = \frac{1}{2}\left(1 - \cos\left(\frac{g^2 n}{\Delta} T_c\right)\right)$$
$$P_e = |c_{e,n}|^2 = \frac{1}{2}\left(1 + \cos\left(\frac{g^2 n}{\Delta} T_c\right)\right)$$

This is for $T_c g^2/\Delta = \pi$ a parity measurement even & odd give up or down. Key: during the measurements the state is PROJECTED. Evident when computing full state & projecting it.

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$\langle \Theta \rangle = \Theta$

$\Theta_0^2 \propto \bar{I}^2$

Signal energy
(power)
bolometer

$\langle \delta\Theta_0^2 \rangle$

$\langle \Delta N^2 \rangle = \bar{N}$

$\Rightarrow$ interval

$\bar{N} \hbar\omega = \frac{1}{K} \overline{b_{in}^\dagger b_{in}}$

$= \frac{\overline{b_{in}}}{K}$

$\bar{N} = \frac{\overline{b_{in}^2}}{\hbar\omega} \frac{1}{K}$

$\langle \Delta N^2 \rangle = \bar{N}$

$\bar{N} = \left(\frac{\overline{b_{in}^2}}{K} \right)$

$\Rightarrow \bar{N} = \frac{\overline{b_{in}}}{K}$

Time measurement!

$\Delta\Theta^2 = \frac{S_{\Theta\Theta}}{t_m} \cdot S_{\alpha\alpha} B$

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$$\langle \Delta\hat{N}^2\rangle = \langle \hat{N}^2\rangle - \langle\hat{N}\rangle^2 = \bar{\alpha}^2 \langle 0|\underbrace{d d^\dagger}_{d^\dagger d+1}|0\rangle = N$$

Define quadratures $\hat{x}_1 = \frac{1}{\sqrt{2}}(\hat{a}+\hat{a}^\dagger)$ $\hat{x}_2 = \frac{i}{2}(\hat{a}^\dagger - \hat{a})$

Let's assume $\langle \hat{x}_1\rangle = \sqrt{2}\alpha$ i.e. $\langle x_1\rangle = \sqrt{2}^{-1}(\alpha+\alpha^*) = \frac{1}{\sqrt{2}}2\alpha = \sqrt{2}\alpha$
(α real) $\langle \hat{x}_2\rangle = 0$ $\left(= \frac{1}{\sqrt{2}}(\alpha-\alpha) = 0\right)$ $\wedge$ $(\langle \hat{x}_2^2\rangle = \frac{1}{2}\langle 0|d d^\dagger|0\rangle)$

Let us define a phase i.e. $\langle\hat{\Theta}\rangle = \frac{\langle \hat{Y}\rangle}{\langle X\rangle}$ $\Theta \ll 1$

Thus $\Delta\Theta^2 = \frac{\langle \hat{x}_1^2\rangle}{(\langle x_2^3\rangle)^2} = \frac{\frac{1}{2}\langle 0|\hat{d}\hat{d}^\dagger|0\rangle}{2\bar{N}} = \frac{1}{4\bar{N}}$

Also note that $\langle \Delta\hat{N}^2\rangle = \bar{N}$

Thus $\boxed{\underbrace{(\Delta\Theta)^2}_{\text{classical "phase" uncertainty}} \cdot \langle\Delta\hat{N}^2\rangle = \frac{1}{4\bar{N}}\cdot\bar{N} = \frac{1}{4}}$ i.e

Returning to the question of measurements. Assume $\langle\sigma_z\rangle = \pm 1$ state of the qubit. $\Theta_\pm = A\cdot\mathrm{arc}\left(\frac{4}{k}\right)\cdot(\pm 1)$ Phase shift of reflected input field

If $\Theta \ll 1$ it will imply that many photons are necessary to determine state of the cavity [weak measurements].
*Note: photon resolving is STRONGLY measured.

The analysis can be repeated for the input fields
$$\hat{a}_{in} = \bar{a}_{in} + \underbrace{\delta\hat{a}_{in}}_{\text{vacuum noise}}$$ $\dot{\bar{N}} = \bar{I} = |\langle\hat{a}_{in}\rangle|^2 = (\bar{a}_{in})^2$

and $\langle \hat{\dot{N}}(t)\hat{\dot{N}}(0)\rangle - \langle\hat{\dot{N}}\rangle = \dot{\bar{N}}\,\delta(t)$ Fluctuations.

Noise spectral density $\underline{S_{\dot{N}\dot{N}} = \dot{\bar{N}}}$ (white) i.e. $S_{\dot{N}\dot{N}} = \int e^{-i\omega t}\underbrace{C(t)}_{\text{auto corrl.}}dt$

Phase $\hat{\Theta} = \frac{i\langle \hat{b}_{in} - \hat{b}_{in}\rangle}{\langle\hat{b}_{in} + \hat{b}_{in}\rangle} = \langle\Theta\rangle + i\,\frac{(\delta a_{in}^\dagger - \delta\hat{a}_{in})}{2\bar{a}_{in}}$

$\Rightarrow$ Noise correlator $\langle \delta\Theta(t)\delta\Theta(0)\rangle = \frac{1}{4\dot{\bar{N}}}\,\delta(t)$ $\Rightarrow S_{\Theta\Theta} = \frac{1}{4\dot{\bar{N}}}$ $\Rightarrow \underline{S_{\Theta\Theta}S_{\dot{N}\dot{N}} = \frac{1}{4}}$

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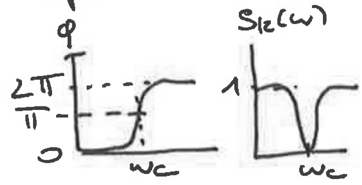
Measurements with parametrically coupled resonant cavities



→ measure phase from cavity

→ weak, continuous measurement

Two examples: • QND measurement for qubit state
• non-QND of mechanical mirror motion

First compute classical phase

$$\hat{H} = \hbar\omega_c \hat{a}^\dagger\hat{a}\,(1 + \underbrace{A\hat{\sigma}_z}_{\text{dispersive shift}}) + \underbrace{\hbar\omega_{12}\hat{\sigma}_z}_{\text{TLS}} + \hat{H}_{diss}$$

The cavities equation of motion with dissipation

(1) $\hat{a}_{out} = \hat{a}_{in} + \sqrt{\kappa}\cdot\hat{a}$; input-output relation

(2) $\partial_t \hat{a} = -i(\omega_c + A\hat{\sigma}_z)\hat{a} - \kappa/2 - \sqrt{\kappa}\,\hat{a}_{in}$

$$\Rightarrow \boxed{r[\omega=\omega_c] = -\left(\frac{1+2iA\hat{\sigma}_z\cdot Q_c}{1-2iA\hat{\sigma}_z Q_c}\right)}$$

(on resonance reflection)

and $Q_c = \omega/\kappa$ "Quality factor"

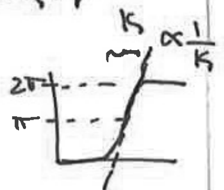
thus $|r(\omega=\omega_c)| = 1$

Phase, $r = |r|e^{i\varphi}\underbrace{e^{i\pi}}_{-1}$ φ: reflected phase.

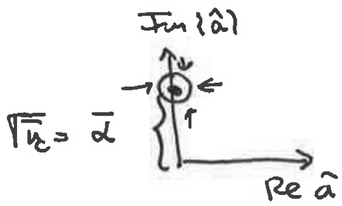
Phase: $\boxed{\varphi \approx 4 Q_c A\hat{\sigma}_z}$

$\tan\varphi = \left(\frac{\mathrm{Re}\{r\}}{\mathrm{Im}\{r\}}\right)^{-1} \approx \varphi$

$$\varphi = 4\frac{\omega_c}{\kappa}A\hat{\sigma}_z = \underbrace{\left(\frac{4}{\kappa}\right)}_{\text{slope of response curve}}\underbrace{\omega_c A\hat{\sigma}_z}_{\text{cavity shift}}$$



next: phasor diagram for the cavity amplitude:



$\hat{a} = \bar{a} + \underbrace{\hat{d}}_{\text{noise}}$ $(\delta\hat{a} \hat{=} d)$ noise operator $[\delta\hat{a}, \hat{d}^\dagger] = 1$

linearization procedure.

$$\bar{N} = \langle\hat{n}\rangle = \langle\hat{a}^\dagger\hat{a}\rangle = \langle 0|(\bar{a}+\hat{d}^\dagger)(\bar{a}+\hat{d})|0\rangle$$
$$= \bar{a}^2 + \bar{a}\langle 0|d^\dagger|0\rangle + \bar{a}\langle 0|d|0\rangle + \langle 0|d^\dagger d|0\rangle$$
$$= \bar{a}^2$$